Qual Review Notes
Linear Algebra & Linear Multistep Review

Kevin LaTourette

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1 The Determinant

- Let $A \in \mathbb{R}^{n\times n}$, and we obtain B by interchanging the rows/column of A, then (*easy to show* 2×2, *use induction for larger systems*)

$$\det B = -\det A \tag{1}$$

- Let $A \in \mathbb{R}^{n\times n}$, and we obtain B by multiplying the i^{th} row/column of A by $k \in \mathbb{R}$, then

$$\det B = k \det A \tag{2}$$

  If we instead take $B = kA$, then

$$\det B = k^n \det A \tag{3}$$

- Let $S_i \in \mathbb{R}^{n\times n}$ be constructed by multiplying the i^{th} row/column of $I^{n\times n}$ by $s \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, then

$$\begin{aligned} \det S_i &= s \\ \det S_i A &= \det S_i \det A \\ &= s \det A \end{aligned} \tag{4}$$

- Let $A \in \mathbb{R}^{n\times n}$. If the rows/columns of A are linearly dependent on the other rows/columns, then A is singular and

$$\det A = 0 \tag{5}$$

- Let $A, B \in \mathbb{R}^{n\times n}$, then

$$\begin{aligned} \det(BA) &= \det B \det A \\ &= \det(AB) \end{aligned} \tag{6}$$

  It should be clear then, that if B were singular, then AB is singular as well.

- Let $A \in \mathbb{R}^{n \times n}$, if A in invertible, then

$$\det A^{-1} = \frac{1}{\det A} \tag{7}$$

- Let $A \in \mathbb{R}^{n \times n}$, then if

$\det A \neq 0$	$\det A = 0$
1. A is non-singular	1. A is singular
2. A is invertible	2. A is non invertible

- Let $A \in \mathbb{R}^{n \times n}$, then

$$\det A^T = \det A \tag{8}$$

- Let $A \in \mathbb{R}^{n \times n}$, and let λ_i be the associated eigenvalues, then

$$\det A = \prod_i \lambda_i \tag{9}$$

2 Factorizations

2.1 SVD

The Singular Value Decomposition. For every $A \in \mathbb{C}^{m \times n}$, there exist unitary matrices $U \in \mathbb{C}^{m \times m}, V \in \mathbb{C}^{n \times n}$ and diagonal matrix $\Sigma \in \mathbb{R}^{m \times n}$ such that

$$A = U \Sigma V^* \tag{10}$$

where the diagonal entries of Σ are $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$. We call $u_i \in U$ the *right singular vectors*, are orthonormal, and $v_i \in V$ the *left singular vectors*, also orthonormal.

2.2 Eigenvalue Decomposition

Let $A \in \mathbb{C}^{n \times n}$, then there may exist $X \in \mathbb{C}^{n \times n}$ whose columns are linearly independent eigenvectors of A, and $\Lambda \in \mathcal{C}^{n \times n}$ is a diagonal matrix whose entries are the eigenvalues of A, we have that

$$A = X \Lambda X^{-1} \tag{11}$$

Not necessarily orthonormal

3 Linear Multi-step Methods

Following the methods of Ascher & Petzold...

For our ODE $y' = f(t, y)$, the general n step linear multistep method is given by

$$\sum_{i=0}^{n} \alpha_i y_{n-j} = h \sum_{i=0}^{n} \beta_j f_{n-j}.$$

We call the method Explicit if $\beta_0 = 0$, implicit otherwise.

3.1 Order & Consistency

A linear mult-step method is order p iff

$$C_0 = C_1 = \ldots = C_p = 0, \quad C_{p+1} \neq 0$$

whose **Local Truncation Error** is given by

$$d_n = C_{p+1} h^p f^{(p+1)}(t_n) + \mathcal{O}(h^{p+1})$$

Where the C_k are given by

$$C_0 = \sum_{i=0}^{n} \alpha_i$$

$$C_j = (-1)^j \left[\frac{1}{j!} \sum_{i=0}^{n} i^j \alpha_i + \frac{1}{(j-1)!} \sum_{i=0}^{n} i^{j-1} \beta_j \right], \quad j = 1, 2, \ldots$$

We call a method **Consistent** if it has order $p \geq 1$, hence we must show that at least

$$\sum_{i=0}^{n} \alpha_i = 0,$$

$$\sum_{i=0}^{n} i\alpha_i + \sum_{i=0}^{n} \beta_i = 0$$

3.2 0-Stability & Convergence

We write the Characteristic polynomials of our linear multi-step method by

$$\rho(\xi) = \sum_{i=0}^{n} \alpha_i \xi^{n-i},$$

$$\sigma(\xi) = \sum_{i=0}^{n} \beta_i \xi^{n-i}$$

Our methods is **0-Stable** iff the roots ξ_i of $\rho(\xi)$ satisfy

$$|\xi_i| \leq 1$$

We then say that our method is **Strongly Stable** if the roots ξ are all contained within the unit ball, with the exception of the root $\xi = 1$. A linear multi-step method is **Weakly Stable** if it is 0-stable but not Strongly Stable.

$$\text{Consistency} + \text{0-Stability} \;\Rightarrow\; \text{Convergence}$$

3.3 Absolute Stability

Note we may write our linear multi-step method may be written, for the problem $y' = \lambda y$

$$\begin{aligned}
\sum_{i=0}^{n} \alpha_i y_{n-i} &= h \sum_{i=0}^{n} \beta_i f_{n-i} \\
\sum_{i=0}^{n} \alpha_i y_{n-i} &= h\lambda \sum_{i=0}^{n} \beta_i y_{n-i}, \quad \text{let } y_n = \xi^n \\
\sum_{i=0}^{n} \alpha_i \xi^{n-i} &= h\lambda \sum_{i=0}^{n} \beta_i \xi^{n-i}
\end{aligned}$$

Or $\rho(\xi) = h\lambda\sigma(\xi)$. Our method is **A-Stable** if the roots ξ_i of $\rho(\xi) = h\lambda\sigma(\xi)$ satisfy $|\xi| \leq 1$. We are essentially finding the values of $z = h\lambda$ such that y_n does not grow as $n \to \infty$. Our method is A-Stable if it's region of Absolute stability is $Re(\lambda) < 0$.

- An explicit multistep method cannot be A-Stable
- The order of an A-Stable linear multi-step method cannot exceed 2

Qual Review Notes

Kevin LaTourette

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- SVD

$$A = \sum_{k=1}^{r} \sigma_k u_k v_k^*$$

  - v_k are eigenvectors of AA^*, u_k are eigenvectors of A^*A, eigenvalues of both are σ_k^2
  - A^*A is Hermitian, so it is unitarily daigonalizable… $\exists$ a unitary V such that $V^*AA^*V =$ diagonal matrix
  - If $A \in \mathbb{R}^{m\times n}$, then AA^* is real symmetric, so it is orthogonally diagonalizable: $\exists$ orthognal R such that $R^TAA^TR =$diagonal matrix
  -

- Def: A square matrix B is called *normal* if $B^*B = BB^*$.

  Theorem 1. *A square matrix B is diagonalizable iff B is normal.*

  - Ex. of normal…
  - Hermitian: $B^* = B$; real symmetric: $B^T = B$
  - Unitary: $B^* = B^{-1}$; orthogonal: $B^T = B^{-1}$
  - Skew-Hermitian: $B^* = -B$; real skew-symmetric: $B^T = -B$

- Jordan form "measures" how close to diagonalizable a matrix is.
  - $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$… not diagonalizable, in Jordan form
  - $x' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x$, $x(0) = x_0$, has solns $x(t) = e^{tA}x_0$, and

$$e^{tA} = \begin{pmatrix} e^{-t} & t \\ 0 & e^{-t} \end{pmatrix}$$

$$- \begin{pmatrix} e^{-t} & t \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ e^{-t} \end{pmatrix}$$

- Let R be real orthogonal. Let u be an eigenvector with associated eigenvalue λ.

$$\begin{aligned} (Ru, Ru) &= (\lambda u, \lambda u) \\ &= |\lambda|^2 \cdot \|u\|^2 \end{aligned}$$

$$\begin{aligned} (Ru, Ru) &= (R^T Ru, u) \\ &= (u, u) = \|u\|^2 \\ &\Rightarrow |\lambda|^2 = 1 \end{aligned}$$

- Find all square roots of $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, Jan 2002, Day 1 #4. How many square roots have all real entries?

  Real symmetric $\Rightarrow$ orthogonally diagonalizable.

$$A = Q \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Q^T$$

$$\begin{aligned} \det(A - \lambda I) &= \begin{pmatrix} \frac{1}{\sqrt{2}} - \lambda & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} - \lambda \end{pmatrix} \\ &= \lambda^2 - 1 \\ &\Rightarrow \lambda = \pm 1 \end{aligned}$$

$$A = Q \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Q^T$$

  Let $B = Q \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm i \end{pmatrix} Q^T \Rightarrow B^2 = Q \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Q^T$

- Some extra stuff. . . Let A be real symmetric. Show eigenvectors are orthogonal, given distinct eigenvalues.

$$\begin{aligned} (Au_1, u_2) - (Au_1, u_2) &= 0 \\ (\lambda_1 u_1, u_2) - (u_1, A^T u_2) &= 0 \\ (\lambda_1 u_1, u_2) - (u_1, A u_2) &= 0 \\ (\lambda_1 u_1, u_2) - (u_1, \lambda_2 u_2) &= 0 \\ (\lambda_1 - \lambda_2)(u_1, u_2) &= 0 \quad , \lambda_1 \neq \lambda_2 \\ &\Downarrow \\ (u_1, u_2) &= 0 \end{aligned}$$

- Big themes... Diagonalizability, Orthogonal Diagonalizability, Jordan Form. Good to know quick examples of different forms. Lookn at simple examples for finding QR, LU...
- $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ Gaussian Elimination fails, so would need to use partial pivoting instead
- $\|A\|_2 = \sigma_1$
- Condition number...
  If A is square and singular, $\kappa_2(A) = \infty$.
  If A is square and non-singular we have $\kappa_2(A) = \frac{\sigma_1}{\sigma_n}$.
  If $A \in \mathbb{C}^{m\times n}$ and A is not of full rank, $\kappa_2(A) = \infty$.
  If $A \in \mathbb{C}^{m\times n}$ and A is of full rank, $\kappa_2(A) = \frac{\sigma_1}{\sigma_n}$

  Remark 1. This depends on if we look at A^*A, or AA^* if not of full rank, since one will be $m \times m$, the other $n \times n$... may lose the zero singular values
- Something else...
$$\lim_{n\to\infty} \int_0^\infty \frac{\sin^{2n}(x)}{1+x^2} dx$$
  $\lim_{n\to\infty} \int_0^\infty \frac{\sin(nx)}{1+x^2} dx$, $f(x) = \frac{1}{1+x^2} \in L^1[0,\infty)$, Riemann-Lebesgue.

Qual Review Notes

Kevin LaTourette

June 6, 2008

- August 1998 Review
- Day 1, Q9 Local Truncation error:

$$\frac{y(t_{n+1}) - y(t_n)}{h} - f(t_n + \theta h, \theta y(t_{n+1}) + (1-\theta) y(t_n))$$

where y is the exact solution of $y' = f(t, y)$. **Trick**: "Center" at $(t_n + \theta h, y(t_n + \theta h))$. Define $t_{n+\theta} = t_n + \theta h$, $y_n = y(t_n)$, $y_{n+\theta} = y(t_n + \theta h)$, $y_{n+1} = y(t_{n+1})$.

$$\begin{aligned}
y_{n+1} &= y_{n+\theta} + y'_{n+\theta}(1-\theta)h + \frac{1}{2} y''_{n+\theta}(1-\theta)^2 h^2 + \ldots \\
y_n &= y_{n+\theta} + y'_{n+\theta}(-\theta)h + \frac{1}{2} y''_{n+\theta}(-\theta)^2 h^2 + \ldots \\
&\quad \text{subtract} \\
y_{n+1} - y_n &= y'_{n+\theta}(1-\theta+\theta)h + \frac{1}{2} y''_{n+\theta}\left[(1-\theta)^2 - \theta^2\right] h^2 + \ldots \\
&= y'_{n+\theta} h + \frac{1}{2} y''_{n+\theta}(1-2\theta)^2 h^2 + \ldots
\end{aligned}$$

What is the difference between $y_{n+\theta}$ and $\theta y_{n+1} + (1-\theta) y_n$?

$$\begin{aligned}
\theta y_{n+1} + (1-\theta) y_n &= \theta \left(y_{n+\theta} + y'_{n+\theta}(1-\theta)h + \frac{1}{2} y''_{n+\theta}(1-\theta)^2 h^2 + \ldots \right) \\
&\quad + (1-\theta) \left[y_{n+\theta} + y'_{n+\theta}(-\theta)h + \frac{1}{2} y''_{n+\theta}(-\theta)^2 h^2 + \ldots \right] - y_{n+\theta} \\
&= y'_{n+\theta}\left[\theta(1-\theta) - (1-\theta)\theta\right] h + \frac{1}{2} y''_{n+\theta}\left[\theta(1-\theta)^2 + (1-\theta)\theta^2\right] h^2 + \mathcal{O}(h^3) \\
&= \frac{1}{2} y''_{n+\theta} h^2 + \mathcal{O}(h^3)
\end{aligned}$$

so

$$\begin{aligned}
f(t_{n+\theta}, \theta y_{n+1} + (1-\theta)y_n) &= f(t_{n+\theta}, y_{n+\theta}) \\
&\quad + f_y(t_{n+\theta}, y_{n+\theta})\left[\frac{1}{2}y''_{n+\theta}(\theta-\theta^2)h^2 + \mathcal{O}(h^3)\right] \\
&= y'_{n+\theta} + f_y(t_{n+\theta}, y_{n+\theta})\frac{1}{2}y''_{n+\theta}(\theta-\theta^2)h^2 + \mathcal{O}(h^3)
\end{aligned}$$

$$\begin{aligned}
\frac{y_{n+1}-y_n}{h} - f(t_{n+\theta}, \theta y_{n+1} + (1-\theta)y_n) = \\
&\left[y'_{n+\theta} + \frac{1}{2}(1-2\theta)y''_{n+\theta}h + \mathcal{O}(h^2)\right] \\
&- \left[y'_{n+\theta} + (\theta-\theta^2)f_y(t_{n+\theta}, y_{n+\theta})h^2 + \mathcal{O}(h^3)\right]
\end{aligned}$$

★

- Trapezoidal method: Average $f(t_n, y_n)$ and $f(t_{n+1}, y_{n+1})$
  Midpoint method: Average y_n and y_{n+1}, then plug into $f \ldots f(t_n + \frac{1}{2}h, \frac{1}{2}[y_{n+1} + y_n])$.

- Day 1, Q12

  Let $f \in \mathcal{D}(\Delta^2)$, $f \neq 0$. WTS $(\Delta^2 f, f)_{L^2} \geq 0$. $\Delta^2 f = \nabla \cdot \nabla\, g$, g defined bleow.

$$\begin{aligned}
(\Delta^2 f, f) &= \int_\Omega (\Delta^2 f) f dv, \quad \text{Let } g = \Delta f \\
&= \int_\Omega (\nabla \cdot \nabla\, g) f dv \\
&= \int_\Omega [\nabla \cdot (f\nabla g) - (\nabla f)\cdot(\nabla g)]\, dv \\
&\Downarrow
\end{aligned}$$

$$\int_\Omega \nabla \cdot (f\nabla g) dv = \oint_{\delta\Omega} (f\nabla g)\cdot \hat{n} dS = 0$$

Since f is 0 on the boudnary. Thus

$$(\Delta^2 f, f) = -\int_\Omega (\nabla f)\cdot(\nabla g) dv$$

Note: $\nabla \cdot (g\nabla f) = (\nabla g)\cdot(\nabla f) + g\Delta f$,
so $(\nabla g)\cdot(\nabla f) = \nabla \cdot (g \nabla f) - g\Delta f$

$$(\Delta^2 f, f) = -\int_\Omega [[\nabla \cdot (g\nabla f) - g\Delta f]\, dv$$

By Divergence Thm

$$\int_\Omega \nabla \cdot (g\nabla f) dv = \oint_{\delta\Omega} g(\nabla f) \cdot \hat{n} dS$$

$$\Downarrow$$

$$\begin{aligned}(\Delta^2 f, f) &= \int_\Omega g\Delta f dv \\ &= \int_\Omega (\Delta f)^2 dv \geq 0\end{aligned}$$

♣

- Day 1, Q11
  Take $f_n(x) = x^n$.
- Day 1, Q8
  $A = \begin{pmatrix} x \\ y \end{pmatrix}$, $|A|^2 = x^2 + y^2$
  $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \mu x - (x^2+y^2)x \\ \mu y - (x^2+y^2)y \end{pmatrix}$
  $(0,0)$ is a fixed pt. Suppose $\mu > 0$, then (x, y) with $x^2 + y^2 = \mu$ is a fixed pt: $(x, y) = (\sqrt{\mu}\cos\theta, \sqrt{\mu}\sin\theta)$.
  $-(|\mu| + (x^2+y^2))x = 0$
  $-(|\mu| + (x^2+y^2))y = 0$

  $$f(x,y) = \begin{pmatrix} \mu x - x^3 - xy^2 \\ \mu y - x^2 y - y^3 \end{pmatrix}$$

  $$\begin{aligned}Df(x,y) &= \begin{pmatrix} \mu - 3x^2 - y^2 & -2xy \\ -2xy & \mu - x^2 - 3y^2 \end{pmatrix} \\ Df(0,0) &= \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}\end{aligned}$$

  try polar coord also

  $$\begin{aligned}\frac{d}{dt} r^2 &= \frac{d}{dt}(x^2+y^2) \\ &= 2(xx' + yy') \\ &= 2\left[(\mu - r^2)x^2 + (\mu - r^2)y^2\right] \\ &= 2(\mu - r^2)(x^2+y^2) \\ &= 2(\mu - r^2)r^2\end{aligned}$$

$$
\begin{aligned}
\frac{d\theta}{dt} &= \frac{xy' - yx'}{r^2} \\
&= \frac{x[\mu - r^2]y - y[\mu - r^2]x}{r^2} = 0
\end{aligned}
$$

- Day 1, Q7. Diverges, use Tonelli's Theorem
- Day 2, Q4

$$[Lu](x) = \lambda x$$

Since $-\partial_{xx} Lu = u$, $-\partial_{xx}$ is the inverse of L. So if u is an eigenfunction of $-\partial_{xx}$, it should be an eigenfunction of L.

$$
\begin{aligned}
Ax &= \lambda x \\
&\Downarrow \\
x &= \lambda A^{-1} x \\
&\Downarrow \\
A^{-1} x &= \frac{1}{\lambda} x
\end{aligned}
$$

$$
\begin{aligned}
[Lu](x) &= \int_{-\pi/2}^{\pi/2} K(x,y)u(y)dy \\
&= \int_{-\pi/2}^{x} \frac{1}{\pi}(\frac{\pi}{2} - x)(\frac{\pi}{2} + y)u(y)dy + \int_{x}^{\pi/2} \frac{1}{\pi}(\frac{\pi}{2} - y)(\frac{\pi}{2} + x)u(y)d
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dx} \int_{-\pi/2}^{x} \frac{1}{\pi}(\frac{\pi}{2} - x)(\frac{\pi}{2} + y)u(y)dy = \\
&\qquad \frac{1}{\pi}(\frac{\pi}{2} - x)(\frac{\pi}{2} + x)u(x) - \int_{-\pi/2}^{x} \frac{1}{\pi}(\frac{\pi}{2} + y)u(y)dy
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dx} \int_{x}^{\pi/2} \frac{1}{\pi}(\frac{\pi}{2} - y)(\frac{\pi}{2} + x)u(y)dy = \\
&\qquad -\frac{1}{\pi}(\frac{\pi}{2} - x)(\frac{\pi}{2} + x)u(x) - \int_{x}^{\pi/2} \frac{1}{\pi}(\frac{\pi}{2} - y)u(y)dy
\end{aligned}
$$

so look for cancellation

$$\frac{d}{dx}[Lu](x) = -\int_{-\pi/2}^{x} \frac{1}{\pi}(\frac{\pi}{2} + y)u(y)dy + \int_{x}^{\pi/2} \frac{1}{\pi}(\frac{\pi}{2} - y)u(y)dy$$

$$\begin{aligned}\frac{d^2}{dx^2}[Lu](x) &= -\frac{1}{\pi}(\frac{\pi}{2} + x)u(x) - \frac{1}{\pi}(\frac{\pi}{2} - x)u(x) \\ &= -u(x)\end{aligned}$$

Let $-\frac{d^2}{dx^2}u = \lambda u$. $u'' + \lambda u = 0$, try $u(x) = e^{\pm i\sqrt{\lambda}x}$, or sines and cosines...

$$u(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$$

Qual Review Notes

Kevin LaTourette

June 13, 2008

- January 1999 Review
- Day 1, Q5

  Set $v = ru$, then get $v_{rr} = v_{tt}$, then take Fourier transform, wrt to r. Thus

$$\begin{aligned}
\widehat{v_{rr}} &= -k^2\widehat{v}, \quad \text{recall} \\
\frac{\partial}{\partial r} &\Leftrightarrow (-ik) \\
\widehat{v_{tt}} &= -k^2\widehat{v} \\
&\Downarrow \\
\widehat{v}(t,k) &= f(k)e^{ikt} + g(k)e^{-ikt} \\
\widehat{v}_t(t,k) &= ik\left[f(k)e^{ikt} - g(k)e^{-ikt}\right] \\
u_t(o,r) &\equiv 0 \\
&\Downarrow \\
v_t(0,r) &\equiv 0 \\
\widehat{v}_t(0,k) &\equiv 0 \\
&= f(k) - g(k) \\
&\Downarrow \\
f(k) &= g(k) \quad \text{then} \\
\widehat{v}(0,k) &= 2f(k) \\
&\Downarrow \\
f(k) &= \frac{1}{2}\widehat{v}_0(k)
\end{aligned}$$

thus

$$\begin{aligned}
\widehat{v}(t,k) &= f(k)\left[e^{ikt} + e^{-ikt}\right] \\
v(t,r) &= \frac{1}{2\pi}\int \hat{v}(t,k)e^{irk}dk \\
&= \frac{1}{2\pi}\int f(k)\left[e^{i(r+t)k} + e^{i(r-t)k}\right]dk \\
&= \frac{1}{2\pi}\int \frac{1}{2}\widehat{v}_0(k)\left[e^{i(r+t)k} + e^{i(r-t)k}\right]dk
\end{aligned}$$

Side note...check out Lebesgue Integration on Eucluidean Space, Frank Jones

$$
\begin{aligned}
v(t,r) &= \frac{1}{2}\left[v_0(r+t)+v_0(r-t)\right] \\
u(t,r) &= \frac{v(t,r)}{r} \\
&= \frac{1}{2r}\left[v_0(r+t)+v_0(r-t)\right] \\
&= \frac{1}{2}\left[u_0(r+t)+u_0(r-t)\right]
\end{aligned}
$$

- Day 1, Q12

  A is real, antisymmetric $\Rightarrow$ A has purely imaginary eigenvalues. Let $A\vec{u_0} = i\lambda\vec{u_0}$, $\lambda \in \mathbb{R}$. Then

  $$
  \begin{aligned}
  \left(I + hA + \frac{1}{2}h^2A^2 + \frac{1}{6}h^3A^3 + \frac{1}{24}h^4A^4\right)\vec{u_0} &= \left(I + ih\lambda - \frac{1}{2}h^2\lambda^2 - \frac{i}{6}h^3\lambda^3 + \frac{1}{24}h^4\lambda^4\right)\vec{u_0} \\
  &\vdots \\
  \vec{u_{n+1}} &= \left[\left(1-\frac{1}{2}h^2\lambda^2\right) + i\left(h\lambda - \frac{1}{6}h^3\lambda^3\right)\right]\vec{u_n}
  \end{aligned}
  $$

  assuming $\vec{u_0}$ is an eigenvector...Need this term above to be $|\cdot| < 1$

  $$
  \left(1-\frac{1}{2}h^2\lambda^2\right)^2 + \left(h\lambda - \frac{1}{6}h^3\lambda^3\right)^2 \leq 1
  $$

  now must solve this ugly thing for h...have fun with that

- Day 2, Q4 Homogeneous eq: $xu'' - u' + (1-x)u = 0$
  One soln: e^x
  Try $u = ve^x$

$$
\begin{aligned}
u'' &= e^x v + 2e^x v' + e^x v'' \\
&= e^x(v'' + 2v' + v) \\
&\text{plug it in,} \\
xu'' - u' + (1-x)u &= xe^x(v'' + 2v' + v) - e^x(v' + v) + (1-x)e^x v \\
&= e^x(xv'' + (2x-1)v') \\
&= 0 \\
xv'' + (2x-1)v' &= 0 \\
v'' + (2 - \frac{1}{x})v' &= 0 \\
v'' &= -(2 - \frac{1}{x})v' \\
\frac{v''}{v'} &= \frac{1}{x} - 2 \\
\ln v' &= \ln x - 2x + c \\
v' &= e^{\ln x - 2x + c} \\
&= axe^{-2x}, \quad a = e^c \\
&\Downarrow \\
v(x) &= (x + \frac{1}{2}e^{-2x} \\
u(x) &= (x + \frac{1}{2})e^{-x}
\end{aligned}
$$

Thus we have our two solutions, then proceed to find a Green's Function...

- Day 2, Q6

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi}} \iint n^2 t(nk) e^{-ikx} \varphi(x) dx dk &= \frac{1}{\sqrt{2\pi}} \int \left[\int n^2 t(nk) e^{-ikx} dk \right] \varphi(x) dx \\
\tilde{k} &= nk \\
&\int n^2 t(\tilde{k}) e^{-i\tilde{k}x/n} \frac{d\tilde{k}}{n} \\
&= n \int_{-1}^{-\frac{1}{2}} 2(-1 - \tilde{k}) e^{-i\tilde{k}x/n} + n \int_{-\frac{1}{2}}^{0} 2\tilde{k} e^{-i\tilde{k}x/n} + n \int_{0}^{\frac{1}{2}} 2\tilde{k} e^{-i\tilde{k}x/n} d\tilde{k} \\
&+ n \int_{\frac{1}{2}}^{1} 2(1 - \tilde{k}) e^{-i\tilde{k}x/n} d\tilde{k}
\end{aligned}
$$

Looking for something like the second derivative...

$$
\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2}
$$

A different approach, using Taylor expansion

$$\begin{aligned}
\widehat{\varphi}(k) &= \widehat{\varphi}(0) + k\widehat{\varphi}'(0) + \mathcal{O}(k^2) \\
\int \widehat{\varphi}(k) t_n(k) dk &= \int \left[\widehat{\varphi}(0) + k\widehat{\varphi}'(0) + \mathcal{O}(k^2)\right] t_n(k) dk \\
&= \widehat{\varphi}(0) \int k t_n(k) dk, \quad \tilde{k} = nk \\
&= 2\left[n^2 \int_0^{\frac{1}{2}} \frac{\tilde{k}}{n} 2\tilde{k} \frac{d\tilde{k}}{n} + n^2 \int_{\frac{1}{2}}^1 \frac{\tilde{k}}{n} 2(1-\tilde{k}) \frac{d\tilde{k}}{n}\right] \\
&\vdots \\
&= \frac{1}{2}
\end{aligned}$$

So our final answer is $\frac{1}{2}\widehat{\varphi}'(0)$

- Day 2, Q7

$$\begin{aligned}
f_n(x) &= \left(\frac{d}{dx} - x\right)^n e^{-x^2/2} \\
&= p_n(x) e^{-x^2/2}
\end{aligned}$$

Polynomials are dense in $L^2(\mathbb{R}, e^{-x^2}dx)$, given $\varphi \in L^2(\mathbb{R}, e^{-x^2}dx)$, there is a polynomial P such that $\|\varphi - P\|_{L^2(\mathbb{R}, e^{-x^2}dx)} < \varepsilon$.
Let $N = \deg(P)$.

$$f_n(x) = p_n(x) e^{-x^2/2}, \quad \text{where } p_n \text{ is an nth degree poly}$$

so $p_1, p_2 \ldots, p_n$ form a linearly independent set.

$$\Downarrow$$

$$P = \sum_{k=1}^{N} a_k p_k$$

so, given $\varphi \in L^2(\mathbb{R}, e^{-x^2}dx)$, and $\varepsilon > 0$, there is an $N \in \mathbb{N}$ and $a_1, a_2, \ldots a_n \in \mathbb{R}$ such that

$$\|\varphi - \sum_{k=1}^{N} a_k p_k\|_{L^2(\mathbb{R}, e^{-x^2}dx)} < \varepsilon$$

then

$$\int_{-\infty}^{\infty} |\varphi(x) - \sum a_k p_k|^2 e^{-x^2} < \varepsilon^2$$

$$\int |\varphi(x)e^{-x^2/2} - \sum a_k p_k(x)e^{-x^2/2}|^2 dx < \varepsilon^2$$

$$\|\varphi(x)e^{-x^2/2} - \sum a_k p_k(x)e^{-x^2/2}\|_{L^2(\mathbb{R})} < \varepsilon$$

but if $\varphi \in L^2(\mathbb{R}, e^{-x^2}dx)$, then $\varphi e^{-x^2} \in L^2(\mathbb{R})$ and vice versa.

$$\int |\varphi(x)|^2 e^{-x^2} dx < \infty \Leftrightarrow \int |\varphi(x)e^{-x^2/2}|^2 dx < \infty$$

Given any $\psi \in \text{Ł}^2(\mathbb{R})$, and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ and $a_1, a_2 \ldots, a_N \in \mathbb{R}$ such that

$$\left\| \psi - \sum_{k=1}^{N} a_k p_k e^{-x^2/2} \right\|_{L^2(\mathbb{R})} < \varepsilon$$

Remark 1. Linear Combinations of $p_n(x)e^{-x^2/2}$ are dense in $L^2(\mathbb{R})$, and

$$\begin{aligned}(p_n(x)e^{-x^2/2}, p_m(x)e^{-x^2/2})_{L^2(\mathbb{R})} &= \int p_n(x)e^{-x^2/2} p_m(x)e^{-x^2/2} dx \\ &= \int p_n(x)p_m(x)e^{-x^2} dx\end{aligned}$$

so...

$$\{f_n(x) = p_n(x)e^{-x^2/2}\}$$

is an orthogonal set in $L^2(\mathbb{R})$, and linear combinations of f_n are dense in $L^2(\mathbb{R})$, therefore $\{f_n\}$ is an orthogonal basis, and $\{\frac{f_n}{\|f_n\|_{L^2}}\}$ is an *orthonormal* basis

- Day 1, Q8

  For an eigenvalue whose geometric multiplicity is equals the algebraic multiplicity, the Jordan Block does not have ones along the super-diagonal...strictly diagonal, otherwise the ones will be present along the super-diagonal

  Now suppose that we don't have access to the complex numbers

  If A is a real matrix, then any complex eigenvalues and eigenvectors come in conjugate pairs.

  If A is real and $\lambda = \alpha + i\beta$ is an eigenvalue with eigenvector u, then $\bar{\lambda} = \alpha - i\beta$ is an eigenvalue with eigenvector $\bar{u}$.

Suppose A is real and $Au = (\alpha + i\beta)u$.

$$\begin{aligned}
A(u_R + iu_I) &= (\alpha + i\beta)((u_R + iu_I) \\
Au_R + iAu_I &= (\alpha u_r - \beta u_I) + i(\beta u_R + \alpha u_I) \\
&\Downarrow \\
Au_R &= \alpha u_R - \beta u_I \\
Au_I &= \beta u_R + \alpha u_I \\
A(u_R \quad u_I) &= (\alpha u_R - \beta u_I \quad \beta u_R + \alpha u_I) \\
&= (u_R \quad u_I) \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}
\end{aligned}$$